Stochastic analysis of a non-normal dynamical system mimicking a laminar-to-turbulent subcritical transition

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The effects of stochastic perturbations on a non-normal dynamical system mimicking a laminar-to-turbulent subcritical transition are investigated both analytically and numerically. It is found that a nonlinear dynamical system with non-normal transient linear growth is very sensitive to the presence of weak random perturbations. The effect of non-normality on the exit probability from the zero fixed point is analyzed numerically for small values of the noise intensity parameter. It is found that an increase in the intensity of the noise, or a decrease of the non-normality parameter leads to qualitative changes in the behavior of the trajectories that can be interpreted as noise-induced phase transitions. By using the Itô formula and the adiabatic elimination procedure a stochastic equation governing the slow evolution of the energy of the non-normal system is derived.

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I. INTRODUCTION

The study of non-normal transient linear growth mechanisms has gained much attention, both experimentally and theoretically, during the past decade, especially after the seminal work of Trefethen *et al.* [1]. The main reason is that this explains the onset of turbulence when the laminar flow passes to a turbulent regime without linear instability [2–6]. Non-normality of the linearized Navier-Stokes evolution operator leads to the transient growth of velocity disturbances, even though the steady mean flow is linearly stable. The nonlinear interactions lead to a further amplification of the initially small but finite disturbances. Nonlinear terms play a vital role in the redistribution of energy to those disturbances which exhibit a linear transient growth. Thus the transition to turbulence is not a consequence of the linear instability of the stationary laminar flow, rather, it is the result of the interaction of the non-normality producing transient amplification of velocity perturbations and energy-conserving nonlinearities driving the system into the basin of attraction of turbulent regime. A comprehensive review of the up-to-date results on such interactions and the resulting onset of shear flow turbulence can be found in the review by Grossmann [7] and recent book by Schmid and Henningson [8].

Several theoretical studies have been devoted to stochastically forced dynamical systems involving a non-normal operator [9,10]. It has been found that these systems have an extraordinary sensitivity to random perturbations, as a consequence, it leads to a great amplification of the variances. However, most research has been focused on linear non-normal systems. The aim of this paper is to study the interaction between the following three factors: nonlinearity, non-normality, and stochastics. In order to gain some insight into this problem, we shall examine the role of external noise in the non-normal dynamical system

$$\frac{du}{dt} = -2\varepsilon u + (u^2+v^2)^{1/2} v,$$

$$\frac{dv}{dt} = -\varepsilon v + u - (u^2+v^2)^{1/2} u, \qquad (1)$$

where ε is a small parameter, chosen in analogy with the inverse Reynolds number. This dynamical system has been suggested by Trefethen *et al.* [1] as a simple model explaining the subcritical transition of the Navier-Stokes equations. It should be noted that several other low-dimensional models have been proposed to explain the onset of a turbulent regime for high Reynolds numbers (see, for example, Refs. [2,4,7]). The dynamical system (1) has three stable equilibrium points including (0,0). The main feature of system (1) is that for $\varepsilon \ll 1$ the linearized evolution operator for the fixed point (0,0) is a highly non-normal matrix that leads to a large transient growth of $v(t)$ prior to an eventual exponential decay. It can easily be found, that for the nonzero initial conditions $u(0)=\varepsilon u_0$ and $v(0)=0$, the solution of the linearized equations is of the form $v(t)=u_0(e^{-\varepsilon t}-e^{-2\varepsilon t})$, $u(t)=\varepsilon u_0 e^{-2\varepsilon t}$. The function $v(t)$ achieves a maximum of order one, on a time scale of order ε^{-1}. Furthermore, although both eigenvalues are negative ($\lambda_1=-\varepsilon;\lambda_2=-2\varepsilon$), a finite fluctuation with exceedingly low amplitude can excite the transition from the fixed point (0,0). The main problem here is to find the minimum amplitude of all fluctuations capable to excite this transition and its dependence on the parameter ε of the form ε^{α}. The threshold exponent α is found to be 3 [1]. This tells us that the basin of attraction of (0,0) shrinks very rapidly as $\varepsilon \to 0$.

II. STOCHASTIC NON-NORMAL DYNAMICAL SYSTEM

One of the purposes of this paper is to understand how random perturbations can affect the dynamics of the non-normal system (1). We simply add two generic uncorrelated Gaussian white noise terms to the right hand side of Eq. (1).

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The dynamical system (1) can then be written in the form of the following stochastic differential equations [11]:

$$du=[-2\varepsilon u+(u^2+v^2)^{1/2}v]dt+(2\delta)^{1/2}dW_1(t),$$

$$dv=[-\varepsilon v+u-(u^2+v^2)^{1/2}u]dt+(2\delta)^{1/2}dW_2(t), \quad (2)$$

where $W_1(t)$ and $W_2(t)$ are the uncorrelated standard Wiener processes. Here we assume for simplicity that the intensity of the noise parameter δ is the same for both stochastic terms.

For deterministic system (1) only small but finite initial perturbations can escape from the basin of attraction for a fixed point at the origin. In this case the main problem is to answer the question "What is the threshold exponent α for transition to turbulence?" [4]. In the stochastic case the key question is "What is the long-time effect of adding noise terms to the nonlinear non-normal dynamical system?" Due to the highly sensitive way that the non-normal systems are affected by random perturbations we can expect that the presence of noise in the right hand side of Eq. (1) may lead to a transition, even for zero initial conditions

$$u(0)=0, \quad v(0)=0. \quad (3)$$

We believe that this is physically significant since in practical situation random fluctuations may often be what induce the subcritical transition. To illustrate the stochastic sensitivity of non-normal system (2), consider its linear approximation

$$du=-2\varepsilon u dt+(2\delta)^{1/2}dW_1(t),$$

$$dv=(-\varepsilon v+u)dt+(2\delta)^{1/2}dW_2(t), \quad (4)$$

with zero initial conditions (3). This is a relatively simple stochastic dynamical system in which the variable $u(t)$ is the Ornstein-Uhlenbeck process with well-known statistical properties, while $v(t)$ is the non-Markov random process whose properties can be easily found [11]. Very important statistical characteristics of system (4) are the second moments, since they mimic the kinetic energy of fluid flow. One can find the following explicit representations for them:

$$m_1(t)\equiv Eu^2(t)=\frac{\delta}{2\varepsilon}(1-e^{-4\varepsilon t}),$$

$$m_2(t)\equiv Eu(t)v(t)=-\frac{2\delta}{3\varepsilon^2}e^{-3\varepsilon t}+\frac{\delta}{2\varepsilon^2}e^{-4\varepsilon t}+\frac{\delta}{6\varepsilon^2},$$

$$m_3(t)\equiv Ev^2(t)=\left(-\frac{\delta}{\varepsilon^3}-\frac{\delta}{\varepsilon}\right)e^{-2\varepsilon t}+\frac{4\delta}{3\varepsilon^3}e^{-3\varepsilon t}-\frac{\delta}{2\varepsilon^3}e^{-4\varepsilon t}+\frac{\delta}{\varepsilon}+\frac{\delta}{6\varepsilon^3}.$$

The limiting values $\overline{m}_i=\lim_{t\to\infty}m_i(t)$ are

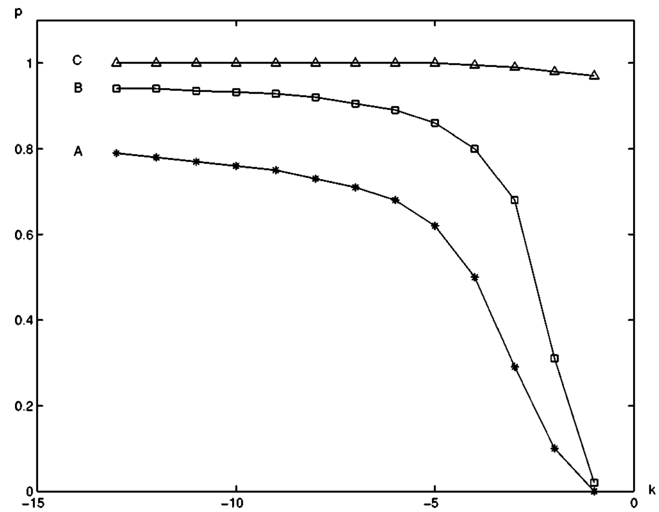

FIG. 1. The exit probability p_e as a function of non-normal parameter $\varepsilon=2^{-k}$. Curves A, B, and C correspond to $\delta=5\times10^{-5}$, 2×10^{-4}, 10^{-3}.

$$\overline{m}_1=\frac{\delta}{2\varepsilon}, \quad \overline{m}_2=\frac{\delta}{6\varepsilon^2}, \quad \overline{m}_3=\frac{\delta}{\varepsilon}+\frac{\delta}{6\varepsilon^3}. \quad (5)$$

From Eq. (5) we can see that due to the non-normality of system (4) as $\varepsilon\to0$ for constant δ, all second moments tend to infinity. The stationary second moment $\overline{m}_3$ exhibits the highest degree of sensitivity. Even for a very weak noise, say $\delta\sim\varepsilon^2$ then $\overline{m}_3\to\infty$ as $\varepsilon\to0$.

It is instructive to investigate the effect that non-normality has on the exit probability from the zero attraction point. This problem is closely related to the famous "Kramer's exit problem" that concerns the escape of random trajectories of a stochastic dynamical system from the domain of attraction of the underlying deterministic dynamical system [11,12]. We have calculated numerically the empirical exit probabilities of random trajectories from the neighborhood of the zero point $U=\{(u,v):u^2+v^2\leqslant0.01\}$ up to $t=10$. The results in Fig. 1, demonstrate that even for a very small intensity of noise ($\delta=10^{-3}$) the exit probability p_e is close to unity. In particular, for $\varepsilon=2^{-5}\approx0.03$ and $\delta=5\times10^{-5}$ the exit probability is greater than 0.6. For $\delta=2\times10^{-4}$ the probability p_e is greater than 0.8. For $\delta=10^{-3}$ this probability is very close to one. Regarding the measure of smallness of δ, it can be any of second moments (5), since they are proportional to δ. It should be noted that it would be difficult to measure the parameter δ directly [13].

An analytical treatment of the stochastic dynamical system (2) is rather difficult, although some approximations are possible, and indeed useful (see Eq. (14) for the slow varying energy of the non-normal system). We have performed simulations of random trajectories of Eq. (2) for different values of ε and δ. Our numerical results show that either by increasing the intensity of noise δ or by decreasing the non-normality parameter ε, stochastic system (2) undergoes a series of phase transitions. We have found three qualitatively different regimes. For a fixed value of ε, this phenomenon can be interpreted as a noise-induced transition. The detailed

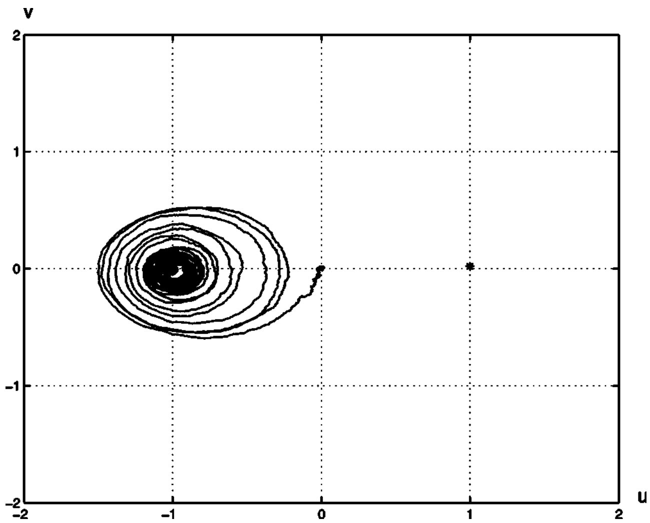

FIG. 2. The stochastic trajectory for $\varepsilon=10^{-2}$, $\delta=10^{-4}$; initial conditions: $u(0)=0$ and $v(0)=0$.

discussion of such transitions and many example from physics, chemistry, biology, etc., can be found in the excellent book [13].

Figures 2 and 3 illustrate these transitions in terms of stochastic trajectories of the non-normal dynamical system for $\varepsilon=10^{-2}$ and $\delta=10^{-4}$ (Fig. 2), $\delta=10^{-2}$ (Fig. 3). For very small values of $\delta(\delta<10^{-12})$, we have observed that the random trajectory is concentrated around the equilibrium point at (0,0). As δ increases, the trajectory then begins to become more concentrated in the vicinity of one of the nontrivial fixed points (Fig. 2). Further increase of the noise intensity parameter δ leads to the stochastic orbits containing all three fixed points (Fig. 3). It should also be noted, that these noise-induced transitions can also be analyzed in terms of the extrema of the stationary probability density $p_{st}(u,v)$ [13]. Figures 4–6 illustrate the qualitative changes in the behavior of the trajectories for the fixed value of $\delta=10^{-3}$ and various values of the non-normal parameter $\varepsilon=10^{-1}$ (Fig. 4), $\varepsilon=10^{-2}$ (Fig. 5), and $\varepsilon=10^{-3}$ (Fig. 6). For a fixed value of the noise intensity parameter δ, one can also speak of non-normality induced phase transition as well.

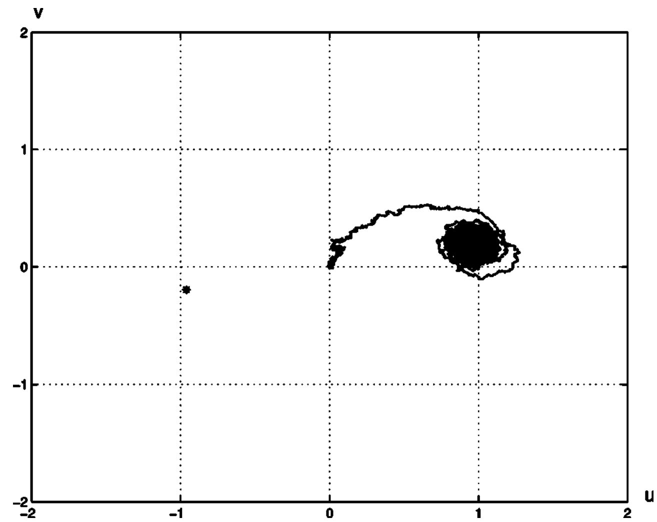

FIG. 4. The stochastic trajectory for $\varepsilon=10^{-1}$, $\delta=10^{-3}$.

III. UNDERLYING HAMILTONIAN STRUCTURE

The behavior of the trajectories when the values of ε and δ are small (e.g., Fig. 6) can be explained by the existence of a Hamiltonian structure in Eq. (2). If we introduce the Hamiltonian function

$$H(u,v)=\frac{1}{3}(u^2+v^2)^{3/2}-\frac{1}{2}u^2, \tag{6}$$

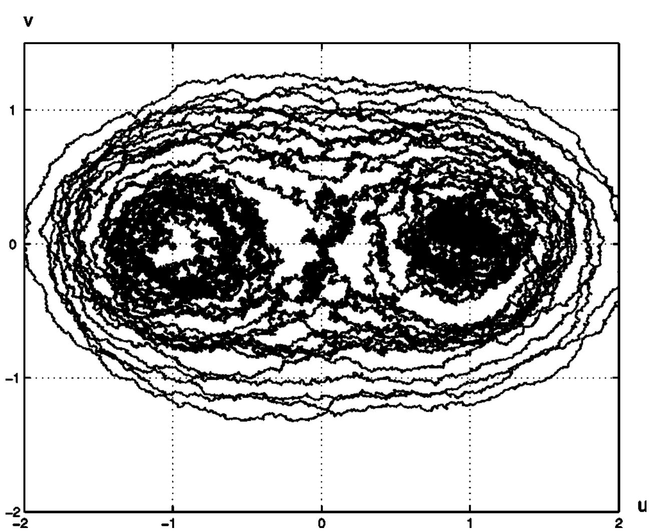

FIG. 3. The stochastic trajectory for $\varepsilon=10^{-2}$, $\delta=10^{-2}$.

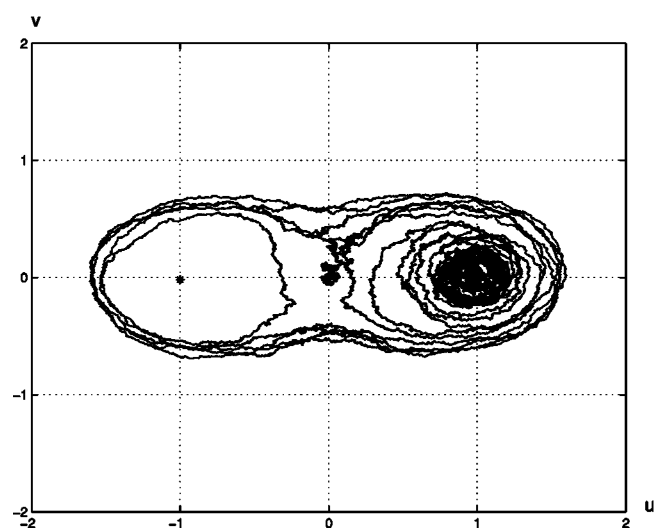

FIG. 5. The stochastic trajectory for $\varepsilon=10^{-2}$, $\delta=10^{-3}$.

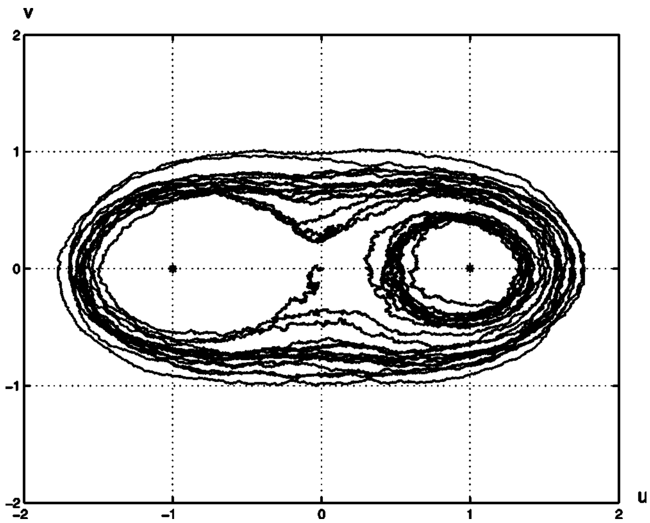

FIG. 6. The stochastic trajectory for $\varepsilon=10^{-3}$, $\delta=10^{-3}$.

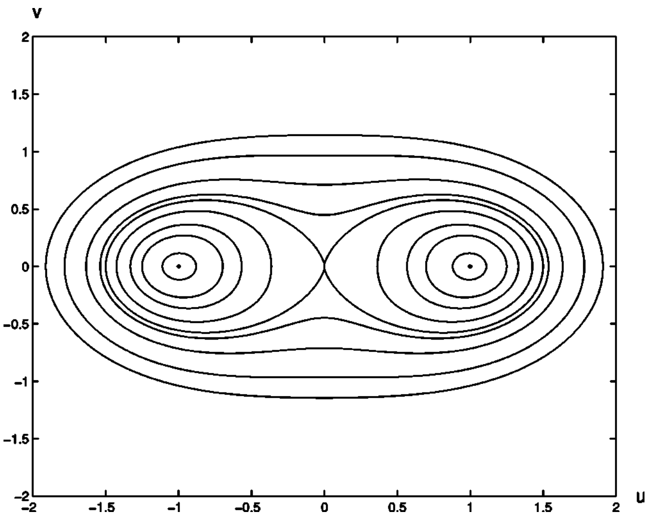

FIG. 7. The phase portrait of the conservative system (7).

then dynamical system (2) can be rewritten as

$$du=-2\varepsilon u\,dt+\frac{\partial H}{\partial v}dt+(2\delta)^{1/2}dW_1(t),$$

$$dv=-\varepsilon v\,dt-\frac{\partial H}{\partial u}dt+(2\delta)^{1/2}dW_2(t). \quad (7)$$

In the limits $\varepsilon\to 0$ and $\delta\to 0$, system (7) becomes conservative,

$$\frac{du}{dt}=\frac{\partial H}{\partial v},\quad \frac{dv}{dt}=-\frac{\partial H}{\partial u}, \quad (8)$$

and, therefore, $H(u,v)=E=$const. The phase trajectories $u(t)$ and $v(t)$ of Eq. (8) move along the level set

$$C(E)=\{(u,v):H(u,v)=\tfrac{1}{3}(u^2+v^2)^{3/2}-\tfrac{1}{2}u^2=E\}, \quad (9)$$

with the speed

$$V(u,v)=\left(\frac{\partial H}{\partial v},-\frac{\partial H}{\partial u}\right). \quad (10)$$

It follows from the existence of the Hamiltonian (6) that the trajectories are periodic, and that the period of the oscillations $T(E)$ can be found to be

$$T(E)=\int_{C(E)}|V(u,v)|^{-1}ds, \quad (11)$$

where the integral is taken along the level curves $C(E)$.

In Fig. 7 we plot the one-parameter family of curves generated by Eq. (9) that gives us the full phase portrait of the conservative system (8). There are three equilibrium points at (0,0) and ($\pm$1,0). One can see that the phase portrait is similar to that of the Duffing equation without dissipation. Linearization of Eq. (8) at (1,0) and ($-$1,0) gives us the period 2π. While moving out, the periodic trajectories have longer periods and tend to infinity as we approach the saddle connection. The situation is more complicated in the presence of dissipative terms. An addition of the two terms $-2\varepsilon u$ and $-\varepsilon v$ changes the direction of the vector field in an alternative way to that of the dissipative Duffing equation. Of course, the global effect is to destroy the closed orbits. In particular, the fixed point (0,0) becomes linearly stable, but the width of its basin of attraction decreases as ε^3 [1,6].

IV. STOCHASTIC DIFFERENTIAL EQUATION FOR THE ENERGY

In a further analysis on the effect of randomness and dissipation, it is of interest to consider the reduced equation for energy. In the general case $(\varepsilon\neq 0,\delta\neq 0)$, the energy of the system $E=H(u,v)$ is not a constant, rather a random function of time. If we apply the Itô formula for $E=H(v,u)$ [11] we can obtain the governing equation for the energy

$$dE=\left(-2\varepsilon\frac{\partial H}{\partial u}u-\varepsilon\frac{\partial H}{\partial v}v+\delta\frac{\partial^2 H}{\partial u^2}+\delta\frac{\partial^2 H}{\partial v^2}\right)dt$$
$$+(2\delta)^{1/2}\frac{\partial H}{\partial u}dW_1(t)+(2\delta)^{1/2}\frac{\partial H}{\partial v}dW_2(t). \quad (12)$$

It is clear that for small values of both the dissipation parameter ε, and the noise parameter δ, after some transient period of time, the phase trajectories of Eq. (7) will be very close to the level curves $C(E)$. There are three different families of periodic orbits separated by the saddle connection (see Fig. 7). Let us denote those components of the level set by $C_i(E)$ $(i=1,2,3)$. The overall dynamics of Eq. (2) can be viewed as a composition of a fast motion along the level curve $C_i(E)$ and of a slow motion normal to the energy levels with the possible transitions, for example, from $C_1(E)$ to $C_3(E)$. In this case one can eliminate the fast motion to derive an equation for the slowly varying energy $E(t)$. It is well known [11] that the fast variables can be eliminated when there exists a stationary distribution function, independent of small parameters. Let us introduce the following normalized measure corresponding to the fast motion [14] along the energy level curve $C_i(E)$:

$$\rho_i(u,v) = \frac{1}{T_i(E)|V(u,v)|}. \tag{13}$$

The equation for the energy $E(t)$ can be derived as follows. Let us multiply Eq. (12) by measure (13), and integrate along the level curve $C_i(E)$ [14]. The equation for $E(t)$ then takes the form of a one-dimensional SDE,

$$\frac{dE}{dt} = S_i(E) - D_i(E) + \sigma_i(E)\frac{dW}{dt}, \tag{14}$$

where the rate of energy supply due to the noise is

$$S_i(E) = \frac{\delta}{T_i(E)}\int_{C_i(E)}\left(\frac{\partial^2 H}{\partial u^2} + \frac{\partial^2 H}{\partial v^2}\right)|V(u,v)|^{-1}ds, \tag{15}$$

while the rate of the removal of energy by dissipation can be written as

$$D_i(E) = \frac{\varepsilon}{T_i(E)}\int_{C_i(E)}\left(2\frac{\partial H}{\partial u}u + \frac{\partial H}{\partial v}v\right)|V(u,v)|^{-1}ds. \tag{16}$$

The intensity of noise is

$$\sigma_i^2(E) = \frac{\delta}{T_i(E)}\int_{C_i(E)}|V(u,v)|ds. \tag{17}$$

The details concerning the derivation of the above formula can be found in Ref. [14]. For very small values of ε and δ most of the probability is concentrated on the level curves $C_i(E)$. We have in essence, a deterministic motion, with speed V along the level curves. In general, we have the stochastically sustained oscillations for which the energy generation $S_i(E)$ due to the noise and dissipation $D_i(E)$ are in balance with the stochastic term, which intensity $\sigma_i(E)$ is a function of energy itself.

An extension of stochastic model (2) to the distributed case involving partial differential equations is now under consideration. We consider the amplification of the mean-field scalar field due to local random perturbations, and subsequent wave propagation into an unstable state [15,16]. The purpose is to find the rate at which the local perturbations propagate throughout the stochastically unstable state.

V. CONCLUSIONS AND DISCUSSION

In summary, we have investigated the effects of the additive Gaussian perturbations on a non-normal dynamical system mimicking a laminar-to-turbulent subcritical transition both analytically and numerically. We have derived explicit representations for the second moments and found that the nonlinear dynamical system with a non-normal transient linear growth is highly sensitive to the presence of weak random perturbations. We have calculated numerically the empirical exit probabilities of random trajectories from the neighborhood of a zero fixed point. We have found that even for very small values of the intensity of noise parameter ($\delta = 10^{-3}$) the exit probability is close to unity. We have also found that an increase of the intensity of noise parameter, or a decrease of the non-normality parameter, will lead to certain qualitative changes in the behavior of the trajectories. This can be interpreted as noise-induced phase transitions. By using the Itô formula and the adiabatic elimination procedure, we have derived a stochastic equation governing the slow evolution of the energy of the system.

We believe that the study of impact of noise on non-normal dynamical system is physically significant since in practical situation random fluctuations may often be what induce the subcritical transition in fluid flow. The transition appears to become essentially random event. The generic feature of laminar-to-turbulent transition in shear flow is that it does not have a critical, reproducible Reynolds number [7,8]. Regarding model (2), it should be noted that its nonlinearity is quite different from the Navier-Stokes one, therefore, it does not really describe the laminar-to-turbulent transition in fluid flow. However, it gives the general features of such transition involving transient growth, nonlinear, and stochastic mode interactions. Stochastic dynamic system (2) is fundamentally different from the deterministic one (1) with only two degrees of freedom. We can regard Eq. (2) as an *effective* dynamical system with many degrees of freedom in which two variables u and v play the role of order parameters, while the stochastic noise terms approximate other degrees of freedom and their influence on u and v. Further research is needed to identify the statistical characteristic of the noise terms in Eq. (2).

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